SKIPPED BLOCKING AND OTHER DECOMPOSITIONS IN BANACH SPACES

STEVEN F. BELLENOT

Abstract. Necessary and sufficient conditions are given for when a sequence of finite dimensional subspaces (X_n) can be blocked to be a skipped blocking decompositon (SBD). These are very similar to known results about blocking of biorthogonal sequences. A separable space X has PCP, if and only if, every norming decomposition (X_n) can be blocked to be a boundedly complete SBD. Every boundedly complete SBD is a JT-decomposition.

1. Introduction

Skipped blocking decompositions (SBD), collections of finite dimension subspaces (X_n) of a Banach space X with additional properties (Definition 2.1), were introduced by Bourgain and Rosenthal [3] to explore RNP, the Radon-Nikodym Property, and the weaker PCP, the point of continuity property. The standard Mazur product construction (Proposition 2.5), basically a global gliding hump, shows that every separable Banach space X has an SBD. This construction has a great deal of flexibility. Properties of the underlining space X can often be transfered to the constructed sequence. For example, Bourgain and Rosenthal [3] showed the existence of a boundedly complete SBD (Definition 5.1) implied PCP. Ghoussoub and Maurey [7] using results form Edgar and Wheeler [5] showed a converse, separable spaces with PCP have a boundedly complete SBD. Other examples are in [14], [6], [1] and [2]. Having a boundedly complete SBD implies the existence of decompositions with stronger properties. Ghoussoub, Maurey and Schachermayer [9] showed such spaces must have a JT-decompostion (Definition 6.1). Like many techniques, SBD existed in the literature before the technique was named. For example, the proof that each separable X has a subspace Y so that both Y and X/Y have FDD [12] (see [13] page 48), is essentially blocking a norming biorthogonal sequence into a SBD.

Our unifying theme is the question: "can one strengthen the *existence* of a (nice) decomposition to the *universal* all decompositions (are nice)?" There is a trivial obstruction, one might need to block the decomposition, the sequence (X_n), in order to make it skipped blocking or make the the decompositon boundedly complete. We introduce the notion of DDD (Definition 2.1) which are the sequences (X_n) that might be blockable to be a SBD. The DDD property is studied without a name in [3] and called a *decomposition* with no adjectives in [14]. Also DDD generalize the well known notation of a biorthogonal sequence. In fact, DDD are nothing more than blockings of biorthogonal sequences. Our general question becomes: "when can every DDD (with perhaps additional properties) be blocked to be a SBD (perhaps with additional properties)?"

The ordering of the sequence (X_n) doesn't matter in the definition of DDD. Thus the collection of DDD include permutations of conditional basic sequences, which can be very ill behaved. Theorem 3.9 states that DDD can be blocked to be a SBD exactly when the predecomposition space $Y \subset X^*$ is c-norming for some c. One can renorm X to improve

to make the c-norming constant 1 (Proprosition 3.13). The space Y is independent of the ordering on (X_n). If (X_n) is DDD, then the dual system (X_n^*) can be blocked to be a SBD (Proposition 3.2).

A number of examples are given that illustrate these results. Example 2.3 shows not every DDD can be blocked to be a SBD. Example 3.4 shows that the dual sequence (X_n^*) of a DDD need not be a SBD. Example 4.2 is a permutation of a basis for James space J and provides a SBD that has badly behaved partial sum projections no matter how the decomposition is blocked. Example 4.4 shows the subspace X_n can be far from the the quotient $X/[X_m]_{m\neq n}$.

We have two complete solutions for the boundedly complete SBD case, or equivalently for separable spaces X with PCP. The point of continuity property, PCP, states every bounded set has a point of weak to norm continuity. Theorem 5.2 shows every norming DDD in a space with PCP can be blocked to be a boundedly complete SBD.

A JT-decomposition (Definition 6.1) is a boundedly complete skipped decomposition with additional properties like those of the predual of JT, James tree space. The fact that PCP implies the existence of a JT-decomposition was proved in [9] using results from sequence of previous papers [7], [8]. Their construction of JT-decomposition added more conditions to the Mazur product construction from the earlier boundedly complete SBD construction. We strengthen this result by showing that each boundedly complete SBD is already a c-norming JT-decomposition (Theorem 6.2) for some $c < \infty$. Theorem 6.3 shows if X has the PCP then every DDD with a c-norming predecompostion spae can be blocked to be a c-norming JT-decomposition and the space X can renormed so that the blocking is a 1-norming JT-decomposition.

The name SBD and its adjectives are somewhat unwieldy. The name boundedly complete SBD applies property "boundedly complete" only to skipped subsequences and not to the global decomposition. The term DDD doesn't demystify a defining-phrase or stand for anything, but we wanted to reserve "decomposition" with no adjective for informal use.

The author would like to acknowledge the help of a referee of an earler version of the paper. For both some connections with biorthogonal sequences and for the examples about total vs norming vs c-norming. These appear at the end of Section 3, starting with Proposition 3.10.

2. Notation, Preliminaries, DDD and SBD

We start the notation about DDD and skipped-blocking decompositions, SBD in this section. Besides notation, we explored the theory with simple examples and observations some of which are known. Also in this section is a well-known preliminary proposition. Every paper on skipped-blocking decompositions seems to have a proof based on the Mazur product construction and Proposition 2.5 is ours.

Our notation generally follows [13] or [10], the first chapter of [11]. In particular, $[X_n]_{n=1}^k$ is the closed linear span of $\cup_{n=1}^k X_n$ and $[X_n] = [X_n]_{n=1}^\infty$. If $m \leq k$ are integers, we will write $X[m,k]$ for $[X_n]_{n=m}^k$ and $X[m,\infty)$ for $[X_n]_{n=m}^\infty$. If $(m(i))$ is a strictly increasing integer sequence with $m(0) = 0$, then we will say $(X[m(i-1)+1, m(i)])$ is a blocking of (X_n). We use the dual pair notation $\langle x, y\rangle$ for $y(x)$ or $x(y)$.

Definition 2.1. *We will say that (X_n), a sequence of finite dimensional subspaces, is a* skipped-blocking decomposition *(*SBD*) for a Banach space X provided (1)–(3) hold. If only (1) and (2) hold we will say (X_n) is a* DDD*.*

(1) $X = [X_n]$. Sometimes this property is called *total*.
(2) For each n, $X_n \cap [X_m]_{m\neq n} = \{0\}$. Sometimes this property is called *minimal*.
(3) For sequences $(n(i))$ and $(m(i))$ with $n(i) < m(i)+1 < n(i+1)$ $(X[n(i),m(i)])_{i=1}^{\infty}$, is FDD for $[X[n(i),m(i)]]$.

Given an DDD (X_n) we define the projections p_n, P_n and R_n as follows:

(4) The projection $p_n : X \to X$ with kernel $[X_m]_{m\neq n}$ and range X_n.
(5) The projection $P_n : X \to X$ given by $P_n = \sum_{i=1}^{n} p_i$.
(6) The projection $R_n : [X_m]_{m\neq n} \to [X_m]_{m\neq n}$ which is the restriction of P_{n-1} or P_n. The more general skipped projections $R_{m,k}$ for $m \leq k$, which is the restriction of P_m on the space $X[1, m-1] \oplus X[k+1,\infty)$. We have $\|R_{n,j}\| \leq \|R_{m,k}\|$ whenever $n \leq m \leq k \leq j$.
(7) The constants $K = \sup\|R_n\|$, $K_\infty = \limsup\|R_n\|$, and $K_{\infty,\infty} = \lim_m \lim_k \|R_{m,k}\|$ Equation (3) is equivalent to $K < \infty$. Equation (2) implies that the projections in (4)–(6) are bounded. The monotone estimate in (6) implies that the limit $K_{\infty,\infty}$ exists.

Definition 2.2. *We will call the constant K in (7), the* SBD-constant*, and the constant K_∞, the* asymptotic SBD-constant*. Sometimes the constants of a* SBD *can be improved by blocking, the constant $K_{\infty,\infty}$ is the* limiting asymptotic constant*, a* DDD *can be blocked to be a* SBD*, if and only if, $K_{\infty,\infty} < \infty$.*

Remark. The sequence (X_n) is an FDD exactly if the projections (P_n) are uniformly bounded which would imply that the projections (p_n) are also uniformly bounded. Conversely, since $P_n = R_{n+1}(I - p_{n+1})$, if (p_n) are uniformly bounded and (X_n) is a SBD, then (X_n) is an FDD. On the other hand, the principal of uniform boundedness says if $\|P_n\|$ is unbounded, then there is an $x \in X$ with $\|P_n x\|$ unbounded. We will see later (Example 4.2), there are SBD (X_n) and x where no subsequence of $(P_n x)$ is bounded. The projections $(P_n x)$ can be very far from x.

Example 2.3. *A* DDD *of one-dimensional subspaces in Hilbert space that is not a* SBD*, nor can it be blocked to be a* SBD*.*

Construction. The sequence (X_n) where $X_n = [e_1 + e_{n+1}/n]$ in Hilbert space with orthonormal basis (e_n) satisfies both (1) and (2) but not (3), so it is a DDD but not a SBD. Since $e_1 + e_{n+1}/n \to e_1$, the sets $[X_m]_{m\neq n} = [e_m]_{m\neq n+1}$ and so the space $X_n^* = [e_{n+1}]$ (see (9) below). No blocking of (X_n) is a SBD, but (X_n^*) is a FDD for its closed linear span. Since $\cap_n X[n,\infty) = [e_1] \neq \{0\}$, the span of (X_n^*) is not all of the dual. Thus the DDD is not separating (Definition 3.8). The projections $p_n(x) = e_{n+1}^*(x)(ne_1 + e_{n+1})$ and hence $p_n(e_1) = 0$ for all n. On the other hand $x = \sum e_{n+1}/n$ has $p_n(x) = e_1 + e_{n+1}/n$ and $\|P_n x\| > n$. Having $\cap_n X[n,\infty) = [e_1] \neq \{0\}$ is the only way a DDD can fail to be a SBD in Hilbert space. □

Example 2.4. *A* SBD *of one-dimensional subspaces in Hilbert space that is not a* FDD*.*

Construction. Let (e_n) be an orthonormal basis, let X_n be the one dimensional $[e_n]$ when n is odd and the one dimensional $[e_{n-1} + e_n/n]$ when n is even. Eventually, $\|p_{2n}\| = 2n$, and $\|R_n\| = 1$. Thus (X_n) is not a FDD but is a SBD with constant one. This also shows even when the X_n are one dimensional in a SBD there need not be a bound on $\|p_n\|$. This is, of course, the standard example that can be found in many places. Clearly the blocking given by $m(i) = 2i$ improves this SBD to a FDD. □

Proposition 2.5. *If X is separable, then X has a* SBD (X_n) *whose asymptotic constant is one.*

Outline of Proof. Let $\varepsilon_n > 0$ so that $\prod(1+\varepsilon_n) < \infty$. Let (x_n) be dense in X. Let $X_1 = [x_1]$ and let W_0 be a finite set of norm one elements of X^* so that $X = X_1 \oplus W_0^\perp$ Inductively pick $(X_n) \subset X$ and finite subsets W_n of the unit sphere of X^* so that

(i) $W_n \supset W_{n-1}$.
(ii) For $x \in X[1,n]$ $\|x\| \le (1+\varepsilon_n)\sup\{|\langle x,y\rangle| : y \in W_n\}$.
(iii) If $x_{n+1} = u + v$ with $u \in X[1,n]$ $v \in W_{n-1}^\perp$ and $v \ne 0$, then there is $y \in W_n$ with $\langle v, y\rangle \ne 0$.
(iv) $X_{n+1} \subset W_{n-1}^\perp$ so that both $W_{n-1}^\perp = X_{n+1} \oplus W_n^\perp$ and $v \in X_{n+1}$. We have $X = X[1, n+1] \oplus W_n^\perp$.

The proof is now clear. But to belabor the point, note (ii) implies $\|R_{n+1}\| \le 1 + \varepsilon_n$ and (iii) implies $X = [X_n]$. □

Example 2.6. *A* SBD *which cannot be blocked to be a* FDD.

Construction. Let X be a Banach space without the Approximation Property, so in particular it cannot have a FDD. Since X has a SBD by Proposition 2.5, this SBD of X cannot be blocked to be a FDD. □

Proposition 2.7. *For any basis (e_i) of X and and permutation π the sequence of one dimensional spaces defined by $X_n = [e_{\pi(j)}]_{j=n}^n$ is a* DDD. *Furthermore, (X_n) is* SBD, *if and only if, $(e_{\pi(n)})$ is a basis.*

Proof. The definition of DDD is invariant under permutations. Since (e_i) is a basis, $x_n = e^*_{\pi(n)}(x)e_{\pi(n)} = p_n(x)$ is uniformly bounded in norm. The remark before Example 2.3 shows if (X_n) is a SBD then (X_n) is an FDD, and hence $(e_{\pi(n)})$ is a basis. The converse is formal. □

3. Duality and the Predecomposition Space

Continuing the list of notation, given an DDD (X_n) we define the quotients Z_n and $Z_{m,k}$ with quotient maps q_n, and $q_{m,k}$ and subspaces X_n^* and Y (the predecomposition space) in the dual, X^*. The quotient maps yield good estimates, unlike the projects (P_n) or (p_n) as Example 4.2 and 4.4 show.

(8) The quotient map $q_n : X \to Z_n = X/[X_m]_{m\ne n}$. And the more general quotients $q_{m,k} : X \to Z_{m,k} = X/(X[1, m-1] \oplus X[k+1,\infty))$. We have $\|q_{m,k}(x)\| \ge \|q_{n,j}(x)\|$ whenever $m \le n \le j \le k$.
(9) The subspaces $X_n^* = [X_m]_{m\ne n}^\perp \subset X^*$. Clearly X_n^* is isometric to Z_n^* via the injection q_n^*. Also X_n^* is the range of the projection p_n^*. Finally, $Z_{m,k}^* = [X_n^*]_{m\le n\le k}$.
(10) $X_n^{*\perp} = [X_m]_{m\ne n}$, for each n.
(11) $X_n^* \cap [X_m^*]_{m\ne n} = \{0\}$, for each n.
(12) p_n^* is a projection on X^* (and Y) with range X_n^* and kernel $X_n^\perp \supset [X_m^*]_{m\ne n}$.

Remark. Each DDD (X_n) is a blocking of a biorthogonal sequence (x_i, x_i^*) obtained by combining the biorthogonal sequences for the finite dimensional space pairs (X_n, X_n^*). The converse is also clear, each grouping of a biorthogonal sequence yields a DDD.

Definition 3.1. *Given a* DDD (X_n) *let* $Y_n = X_n^*$ *and let* $Y = [Y_n] = [X_n^*] \subset X^*$*. We will say* Y *is the predecomposition space of* (X_n)*. Clearly* (Y_n) *is a* DDD *for* Y *by equation (11). Note that (9) implies that any blocking of* (X_n) *has the same predecomposition space* Y*.*

Proposition 3.2. *Any* DDD (X_n) *can be blocked to* $(X[m(i-1)+1, m(i)])$ *so that resulting dual* DDD $(Y[m(i-1)+1, m(i)])$ *is a* SBD *for the predecomposition space* Y*.*

Proof. Suppose (X_n) is a DDD. Let $\varepsilon > 0$ be given with $\varepsilon < 1$. Once $m(k)$ is selected, consider the quotient $Z = Z_{1,m(k)}$ and subspace of Y given by $W = q^*_{1,m(k)}(Z^*)$. We can find a finite subsets $\{z_i\}_1^N$ and $\{w_i\}_1^N$ of the unit sphere of Z and W so that $\langle z_i, w_i\rangle = 1$ and $\{w_i\}_1^N$ is an ε-net for the sphere of W. Since $\operatorname{span} \cup X_n$ is dense we can find x_i so that $1 \le \|x_i\| < 1+\varepsilon$ and $q_{1,m(k)}(x_i) = z_i$. Finally select $m(k+1)$ so that $\{x_i\}_1^N \subset X[1, m(k+1)]$.

So if $y \in [X_n^*]_{n>m(k+1)}$ and w is in the sphere of W, then by the construction above we can find w_i with $\|w - w_i\| < \varepsilon$ and clearly, $y(x_i) = 0$. Hence $\|w + y\| \ge \|w_i + y\| - \|w - w_i\|$ and $\|w_i+y\| \ge |(w_i+y)(x_i)|/\|x_i\| \ge 1/(1+\varepsilon)$. So we have $\|w+y\| \ge (1-\varepsilon)(1+\varepsilon)^{-1}\|w\|$ and the $R_{m(k)+1,m(k+1)}$ projection onto $[X_n^*]_{n\le m(k)}$ with kernel $[X_n^*]_{n>m(k+1)}$ has norm bounded by $(1 + \varepsilon)/(1 - \varepsilon)$. ☐

Question 3.3. *If* (X_n) *is a* SBD*, is the dual* DDD (X_n^*) *already a* SBD*?*

Example 3.4. *A* DDD (X_n) *for* ℓ_2*, where* (X_n^*) *is not a* SBD *for* $Y = [X_n^*] = \ell_2$*.*

Construction. Let (e_i) be a conditional basis for ℓ_2 and let π be a permutation so that $(e_{\pi(n)})$ is not a basis. Thus the coefficient functionals $(e^*_{\pi(n)})$ are also not a basis. Letting $X_n = [e_{\pi(j)}]_{j=n}^n$, Proposition 2.7 says (X_n^*) is not a SBD. ☐

Corollary 3.5. *For any basis* (e_n) *and permutation* π*, the sequence* $(e_{\pi(n)})$ *can be blocked to be a* SBD*.*

Proof. Apply Proposition 3.2 to the coefficient functions $(e^*_{\pi(n)})$. ☐

Lemma 3.6. *Suppose* (X_n) *is a* DDD*, then the following estimates hold:*

(a) *If* $w \in X[1, k-1]$ *and* $j \ge k$*, then* $\|w\| \le \|R_{k,j}\|\|q_{1,j}(w)\|$ *and in particular,* $\|w\| \le \|R_k\|\|q_{1,k}(w)\|$*.*

(b) *If* $x \in X$*, and* $\delta_k = \operatorname{dist}(x, X[1, k-1])$ *then* $\|x\| \le \|R_k\|\|P_k x\| + \delta_k(1 + \|R_k\|)$*.*

(c) *If* (X_n) *is a* SBD*, then for all* $x \in X$*,*

$$\|x\| \le K_\infty \limsup \|P_n x\| \text{ and } \|x\| \le K \liminf \|P_n x\|$$

.

(d) *If* $w \in X[1, k-1]$*, and* $j \ge k$*, then there is* $y_j \in [X_i^*]_1^j$ *with* $\|y_j\| = 1$ *and* $\|w\| \le \|R_{k,j}\||\langle w, y_j\rangle|$*.*

(e) *The predecomposition space* Y *separates points in* X*, if and only if,* $\cap_n X[n, \infty) = \{0\}$*.*

Proof. Parts of this proof are essentially stolen from the proofs of Lemma I.15 and Lemma I.12 of [9].

Let $w \in X[1, k-1]$ and let $z \in X[j+1, \infty)$, and note $\|w\| = \|R_{k,j}(w+z)\| \le \|R_{k,j}\|\|w+z\|$. Hence $\|w\| \le \|R_{k,j}\|\|q_{1,j}(w)\|$ which proves (a).

Let $x \in X$, $\varepsilon > 0$ and $\delta_k = \operatorname{dist}(x, X[1, k-1])$. Find $w \in X[1, k-1]$ with $\|w-x\| < \delta_k+\varepsilon$. We have $w = P_k w$, $q_{1,k}(x) = q_{1,k}(P_k x)$ and hence $\|q_{1,k}(w) - q_{1,k}(P_k x)\| = \|q_{1,k}(w - x)\| \le$

$\delta_k + \varepsilon$. Putting the pieces together,

$$\|q_{1,k}(P_k x)\| + \delta_k + \varepsilon \geq \|q_{1,k}(w)\| \geq \|w\|/\|R_k\|$$
$$\|R_k\|\|P_k x\| + \|R_k\|(\delta_k + \varepsilon) \geq \|w\| \geq \|x\| - \delta_k - \varepsilon$$

Which completes (b).

If (X_n) is not a SBD, then the inequalities in (c) are ambiguous, as Example 2.3 shows that $K = K_\infty = \infty$ and $P_k x = 0$ for some x and all k. The usual convention $0 \cdot \infty = 0$ gives the wrong result for general DDD. Since δ_k is monotonically decreasing to zero we can derive the first part of (c) from (b) by picking k with $\|R_k\| \approx \limsup \|R_k\|$. We derive the second part of (c) from (b) by picking k with $\|P_k x\| \approx \liminf \|P_k x\|$.

For (d) one uses (a) and the fact $Z^*_{1,j} = [X^*_i]^j_1$ isometrically. So there is a $y_j \in Z^*_{1,j}$ with $\|y_j\| = 1$ and $\langle w, y_j\rangle = \|q_{1,j}(w)\|$.

The statement (e) is almost immediate. If $x \notin X[n+1, \infty)$, then $q_{1,n}(x) \neq 0$. So there is a $x^* \in Z^*_{1,n} \subset Y$ with $\langle x, x^*\rangle \neq 0$. Thus Y separates the points of X when $\cap_n X[n, \infty) = \{0\}$. Conversely, if there is a non-zero $x \in \cap_n X[n, \infty)$ then $\langle x, x^*\rangle = 0$ for $x^* \in X^*_m$ and any m. So x is zero on a dense subset of Y. Thus Y doesn't separate the point x from 0. □

Definition 3.7. *Let $c < \infty$, space $Y \subset X^*$ is said to c-norms X, if all $x \in X$,*

$$c^{-1}\|x\| \leq \|x\|_Y = \sup\{|\langle x, y\rangle| : y \in Y, \|y\| \leq 1\} \leq \|x\|$$

Definition 3.8. *A* DDD *is said to be* separating*, if the predecompostion space Y separates the points of X or equivalently if $\cap X[n, \infty) = \{0\}$.*

A DDD *is said to be norming, if the predecomposition space Y is c-norming for some $c < \infty$.*

Theorem 3.9. *If (X_n) is a* DDD *then the predecomposition space Y c-norms X for $c = K_{\infty,\infty}$. Conversely if for some $c < \infty$, Y c-norms X then the* DDD *can be blocked to be a* SBD *with asymptotic constant $\leq c$.*

Proof. Let $\varepsilon > 0$ be given. If $K_{\infty,\infty} = \infty$ there is nothing to prove. Otherwise, by blocking (X_n) by $m(i)$ we can assume $\|R_{m(i-1)+1,m(i)}\| < K_{\infty,\infty} + \varepsilon$. Lemma 3.6d shows Y $K_{\infty,\infty} + \varepsilon$-norms X. Since Y is independent of the blocking, and ε is arbitrary, Y c-norms X.

To show the converse, we need a blocking $(m(i))$ so that the projections $R_{m(i-1)+1,m(i)}$ are uniformly bounded. Let $m(0) = 0$, $m(1) = 1$ and suppose $m(k)$ has been selected. Let $\varepsilon > 0$ be given. We can find unit vectors w_j which are an ε-net in the sphere of $X[1, m(k)]$. We can find unit vectors y_j in Y so that $|\langle w_j, y_j\rangle| > c^{-1} - \varepsilon$. Using the denseness of $\operatorname{span} \cup X^*_n$ in Y, select $v_j \in \operatorname{span} \cup X^*_n$ so that $\|v_j - y_j\| < \varepsilon$. Pick $m(k+1)$ so that each $v_j \in [X^*_n]^{m(k+1)}_1$. Let $z \in X[1, m(k)]$ have norm one, select j so that $\|z - w_j\| < \varepsilon$. We have

$$\langle z, v_j\rangle = \langle w_j, y_j\rangle + \langle z - w_j, y_j\rangle + \langle z, v_j - y_j\rangle$$
$$|\langle z, v_j\rangle| > c^{-1} - 3\varepsilon$$

And since $\langle u, v_j\rangle = 0$ if $u \in X[m(k+1) + 1, \infty)$, $\|z + u\| > (c^{-1} - 3\varepsilon)\|z\|$ and so $\|R_{m(i-1)+1,m(i)}\| < c/(1 - 3\varepsilon c)$. □

Proposition 3.10. *If X is separable and Y a separating subspace in X^*, then there is a* DDD *(X_n) with predecomposition space Y.*

Proof. The usual construction of a biorthogonal sequence (x_n, x^*_n) ([13] page 43), is flexible enough to produce sequences so that $X = [x_n]$ and $Y = [x^*_n]$. Then letting $X_n = [x_i]_{i=n}$, we have a DDD of one dimensional spaces with predecomposion space Y. □

Example 3.11. *There is a separating* DDD *that can't be blocked to be a* SBD*.*

Construction. Each non-quasi-reflexive space X has a separable subspace Y of the dual which is separating but non-norming [10]. □

Example 3.12. *There is a c-norming predecomposition space Y that isn't one 1-norming.*

Construction. Let the norm one $\phi \in X^{**}$ satisfy $0 < \operatorname{dist}(\phi, B_X) < 1$ where B_X is is the unit ball of X as a subspace of X^{**}. Let $Y = \ker \phi \subset X^*$. If x has norm one and $\|x - \phi\| = \gamma < 1$, for norm one $x^* \in Y$ $|x^*(x)| = |x^*(x - \phi)| \leq \gamma < 1$, so at best Y can only γ^{-1}-norm X. □

Proposition 3.13. *If* DDD (X_n) *can be blocked to be a* SBD *for X, then in the equivalent norm $\|\cdot\|_Y$, (X_n) is still a* DDD *with the same predecompostion space Y but now it is 1-norming and (X_n) can be blocked to have asymptotic constant 1.*

Proof. The condition on (X_n) implies Y c-norms X for some $c < \infty$. Thus $\|\cdot\|_Y$ is an equivalent norm on X. Clearly being a DDD or a SBD is invariant under isomorphic norms. Since

$$\|x\|_Y = \lim_k \|q_{1,k}(x)\| = \lim_k \operatorname{dist}(x, X[k+1, \infty))$$

The quotient spaces $Z_{1,k}$ obtained for the original norm and the new $\|\cdot\|_Y$ are isometric. It follows that the predecomposition spaces are identical and the norms are isometric. Clearly Y 1-norms $(X, \|\cdot\|_Y)$. The result now follows from Theorem 3.9. □

4. Structure of SBD

Basically this section shows the relationship between a SBD (X_n) and X is not very strong in general. The quotient spaces (Z_n) might have more importance.

Proposition 4.1. *If (X_n) is a* DDD *for X and $x \in X$, then there is a subsequence $(m(i))$ and $w_i \in X[m(i-1)+1, m(i)]$ so that $P_{m(i)}x + w_{i+1} \to x$.*

Proof. Let $x \in X$ be given and suppose (X_n) is DDD and $m(k)$ has been found. Let $M = \|P_{m(k)}\|$. We can find $w \in \operatorname{span} \cup X_n$ so that $\|x - w\| < 2^{-k}/M$. We have $\|P_{m(k)}x - P_{m(k)}w\| < 2^{-k}$ and so we can let $w_{k+1} = w - P_{m(k)}w$ and

$$\|x - (P_{m(k)}x + w_{k+1})\| \leq \|x - w\| + \|P_{m(k)}w - P_{m(k)}x\| < 2 \cdot 2^{-k}.$$

Finally we select $m(k+1)$ large enough so that $w_{k+1} \in X[m(k)+1, m(k+1)]$. □

Example 4.2 shows that $(\|P_{m(i)}x\|)$ and hence $(\|w_i\|)$ can be unbounded even when the DDD is a SBD.

Example 4.2. *A* SBD *of James space J and a x so that no subsequence of $(P_n x)$ is bounded.*

Construction. James space J, can be written as the set of null sequences (a_n) with finite norm given by $\|(a_n)\|^2 = \sup \sum |a_{n(i+1)} - a_{n(i)}|^2$, over all increasing finite integer sequences $(n(i))$. The usual unit basis (e_n) is a shrinking conditional basis which is not boundedly complete. Let (f_n) be the coefficient functionals. Let $w_k = \sum\{e_n : 2^{k-1} < n \leq 2^k\}$ and let $x = e_1 + \sum_{k>0} w_k/k$. Eventually, $\|x\| = 1$.

Let π be the permutation that reorders the integers in alternating blocks of evens and odds in the order:

$$1, 2, B_2, B_3, A_2, B_4, A_3, \ldots, B_k, A_{k-1}, B_{k+1}, A_k \ldots$$

Where $A_k = \{n \text{ odd} : 2^{k-1} < n \le 2^k\}$ and $B_k = \{n \text{ even} : 2^{k-1} < n \le 2^k\}$. Let (X_n) be any blocking of $([e_{\pi(i)}])$ that is a SBD.

Let's estimate the norm of $z_n = \sum_1^n f_{\pi(i)}(x)e_{\pi(i)}$ and assume $\pi(n)$ is in A_{k-1} or B_{k+1}. This means $f_j(z_n) = 0$ for $j \in A_k$ and $f_j(z_n) = 1/k$ for $j \in B_k$. Thus $\|z_n\| \ge \sqrt{2^k/k^2}$. It follows that every subsequence of $(P_n x)$ is unbounded no matter how $([e_{\pi(i)}])$ was blocked. □

Lemma 4.3. *If (X_n) is a DDD and $x \in X$, then $\lim q_n(x) = 0$*

Proof. Let $x \in X$ and $\varepsilon > 0$. Since span (X_n) is dense in X we can find N and $y \in X[1, N]$ so that $\|x-y\| < \varepsilon$. Since $x-(x-y) = y$ is in $X[1, N]$, $q_n(x) = q_n(x-y)$ and $\|q_n(x-y)\| < \varepsilon$ for $n > N$. □

Example 4.4. *A SBD (X_n) and $x \in X$ so that $(p_n(x))$ has no bounded subsequence. In particular, the isomorphisms $q_n|_{X_n} : X_n \to Z_n$ have inverses with norms that blow up. Thus X_n and Z_n can be far apart.*

Construction. This is a continuation of Example 4.2. For $k > 3$, let $X_k = [e_i : i \in B_k \cup A_{k-1}]$. The same x, has $\|p_k(x)\| \ge \sqrt{2^k/k^2}$, while $q_k(x) = q_k(p_k(x)) \to 0$ by the lemma. □

5. Y^* and Boundedly Complete SBD

Definition 5.1. *If $x_n \in X_n$ and $(\|\sum_1^n, x_i\|)$ bounded implies that $\sum x_i$ converges in norm, then we call (X_n) is boundedly complete. A boundedly complete SBD is one where all the skipped decompositions (X[n(i),m(i)]) in equation (3) are boundedly complete. (Note that the whole sequence (X_n) is not required to be boundedly complete.)*

The subspace map $\phi : Y \to X^*$ yields by duality a quotient map $\phi^* : X^{**} \to Y^*$. The restriction of ϕ^* to the image of X in X^{**} is the duality given by $\langle y, \phi^* x\rangle = \langle x, y\rangle$ for $y \in Y$. The adjoint of quotient maps $q_{m,k} : X \to Z_{m,k}$ given by (7), factors through Y as the subspace injections $Z^*_{m,k} \subset Y \subset X^*$ so the double adjoint quotient map $q^{**}_{m,k} : X^{**} \to Z^{**}_{m,k} = Z_{m,k}$ factors through Y^*. Similarly we can identify X_n in Y^* (as a set with perhaps a different norm) as the range of the projection $p^{**}_n = \phi^*(X_n)$. Note that ϕ^* is an isomorphism exactly when Y is norming. Theorem 3.9 implies ϕ^* is an isomorphism exactly when the DDD (X_n) can be blocked to be a SBD. Note further that ϕ^* is an isometry if X has PCP or if (X_n) is shrinking.

On the unit ball, B_{Y^*}, of Y^* there are several differently defined weak topologies which are the same. There is $\sigma(X^{**}, X^*)$-topology on $B_{X^{**}}$ which the quotient map ϕ^* induces on B_{Y^*} which must be the $\sigma(Y^*, Y)$- topology by compactness and weak-star continuity. Also this is the same as the $\sigma(Y^*, \cup Y_n)$, also because of compactness. In this topology, B_{Y^*} is a compact metric space.

For G a finite subset of $\cup Y_n$ and $\varepsilon > 0$, let

$$V(x, G, \varepsilon) = \{y^* \in Y^* : | < y^* - x, g > | < \varepsilon, \forall g \in G\}.$$

it follows that the collection of $V(x, G, \varepsilon)$ is a basis for the $\sigma(Y^*, Y)$-topology on Y^*. We use the term *elementary* to describe $\sigma(Y^*, Y)$-open sets of the form $V(x, G, \varepsilon)$ with $G \subset \cup Y_n$.

Theorem 5.2. *If X has PCP then each norming DDD can be blocked to be a boundedly complete SBD.*

Proof. Let (X_n) be a DDD with a c-norming predecomposition space for X. Renorming via Proposition 3.13 if necessary, we can assume the predecomposition space Y is 1-norming.

By blocking if necessary we can assume that (X_n) is already a SBD. We pick $(m(k))$ by induction so that $m(0) = 0$ and $(X[m(k-1)+1, m(k)])$ is a boundedly complete SBD. Our mode of proof is to make slight modifications to the existence proof in [7].

The PCP property (Lemma II.3 [7]) implies that $Y^* \backslash X = \cup K_n$ for some increasing sequence (K_n) of $\sigma(Y^*, Y)$-compact sets. While the cited lemma constructs a particular Y, any 1-norming subspace of X^* will satisfy the proof.

Suppose $m(k)$ has been selected. For each x in the unit ball of $X[1, m(k)]$ there is an elementary $\sigma(Y^*, \cup Y_n)$-open V with $V \cap K_{k+1} = \emptyset$. By compactness we can assume only a finite number of the open sets $V_i = V(x_i, G_i, \varepsilon_i)$ are needed. Let $m(k+1)$ be large enough so that $Y[1, m(k+1)]$ contains all the vectors in G_i and also $1 + 1/k$-norms $X[1, m(k)]$.

The claim is that the SBD $(X[m(k-1)+1, m(k)])_k$ is also a boundedly complete SBD. Suppose $x_k \in X[m(k-1), m(k)]$ with $x_k = 0$ infinity often. Let $n(i)$ be the sequence defined so that $x_{n(i)} \neq 0$ but $x_{n(i)+1} = 0$ and $s_i = \sum_1^{n(i)} x_k$ be so that $\|s_i\| \leq 1$ uniformly. Let $s \in Y^*$ be the $\sigma(Y^*, Y)$-limit of (s_i).

If $s \in Y^* \backslash X$ then $s \in K_j$ for $j \geq N$ for some N. Let $k = n(i) > N$ so that $x_{k+1} = 0$. Then there is $V = V(w, G, \varepsilon)$ so that $s_i \in V$ and $V \cap K_{k+1} = \emptyset$. Since $G \subset Y[1, m(k+1)]$, $< x_i, g >= 0$ for $i > k+1$. Since $x_{k+1} = 0$ This means $s_j \in V$ for all $j \geq i$, so $s \in V \cap K_{k+1}$ a contradiction.

Therefore $s \in X$, and we need to show s_i converges in norm to s. Let $\varepsilon > 0$ be given. Find $k = n(j)$ and $w \in X[1, m(k)]$ so that $\|s - w\| < \varepsilon$ and $x_{k+1} = 0$. For $i \geq j$ Find $y \in Y[1, m(n(j)+1)]$ with $\|y\| = 1$ and y almost norms $s_i - w$. In particular $\langle y, s_i - w \rangle \leq \|s_i - w\|(1 + 1/i)^{-1}$. Since $\langle y, x_n \rangle = 0$ for $n > n(i)$, we have $\langle y, s - w \rangle = \langle y, s_i - w \rangle$. Thus $\|s_i - w\| \leq \|s - w\|(1 + 1/i)$ and $\|s_i - s\| \leq 3\varepsilon$. Thus (s_i) norm converges to s. □

6. JT-decompositions

Definition 6.1. *The boundedly complete* SBD (X_n) *is called a* c-norming-JT-decomposition *for* X *provided*

(A) *For each* $x^{**} \in X^{**}$, *with* $\liminf \|q_n^{**}(x^{**})\| = 0$, *there is* $x \in X$ *with* $\|x\| \leq c\|x^{**}\|$ *and for all* n, $p_n^{**}(x^{**}) = p_n(x)$.

Ghoussoub, Maurey and Schachermayer [9] have defined a JT-decomposition to be equivalent to what we have called a 1-norming-JT-decomposition. Furthermore they show each separable Banach space with PCP has a 1-norming-JT-decomposition. The following proposition shows every boundedly complete SBD is a c-norming-JT-decomposition.

Proposition 6.2. *If* (X_n) *is a boundedly complete* SBD *for* X, *then* (X_n) *is a* K_∞*-norming-*JT*-decomposition where* K_∞ *is the asymptotic-*SBD*-constant of* (X_n).

Proof. First note that if $x \in X \backslash \{0\}$, then Lemma 3.6c implies for some n, $p_n(x) \neq 0$. It follows that if such an $x \in X$ as in (A) exists, then it must be unique.

Let $x^{**} \in X^{**}$ and let $\varepsilon > 0$ be given. Find $m(1)$ so that $\|q_{m(1)}^{**}(x^{**})\| < \varepsilon/2^1$ and let $w_1 \in \operatorname{span} \cup X_n$ so that $q_{m(1)}(w_1) = q_{m(1)}^{**}(x^{**})$ and $\|w_1\| < \varepsilon/2^1$. Suppose $(w_i)_{i=1}^k$ and $(m(i))_{i=1}^k$ have been chosen so that

(i) $q_{m(i)}(w_i) = q_{m(i)}^{**}(x^{**})$;
(ii) $\|q_{m(i)}(w_i)\| < \varepsilon/2^i$;
(iii) $q_{m(j)}(w_i) = 0$ for $j < i \leq k$;
(iv) $w_{i-1} \in X[1, m(i)-1]$ for $i \leq k$; and

(v) $w_k \in \operatorname{span} \cup X_n$.

Let $C = \|I - p_{m(1)}\| \, \|I - p_{m(2)}\| \cdots \|I - p_{m(k)}\|$ and pick $m(k+1)$ large so that $w_k \in X[1, m(k+1)-1]$ and $\|q^{**}_{m(k+1)}(x^{**})\| < \varepsilon/C2^{k+1}$. Find $z \in \operatorname{span} \cup X_n$ so that $\|z\| < \varepsilon/C2^{k+1}$ and $q_{m(k+1)}(z) = q^{**}_{m(k+1)}(x^{**})$. Let $w_{k+1} = (I - p_{m(1)})(I - p_{m(2)}) \cdots (I - p_{m(k)})z$. Clearly $\|w_{k+1}\| < \varepsilon/2^{k+1}$ and $q_{k(i)}(w_{k+1}) = 0$ for $i \leq k$. Therefore, writing $w = \sum w_k$,

(B) For each $\varepsilon > 0$ there is $w \in X$ and $(m(i))$ so that $\|w\| < \varepsilon$ and $q_{m(i)}(x^{**} - w) = 0$ for all i.

Now for each i, $x^{**} - w \in [X_n]^{\perp\perp}_{n \neq m(i)} = X^{*\perp}_{m(i)}$, hence

$$\| \sum_{j=1}^{m(i)-1} p^{**}_j (x^{**} - w) \| = \|R_{m(i)}(x^{**} - w)\| \leq \|R_{m(i)}\| \, \|x^{**} - w\| \leq K\|x^{**} - w\|.$$

Since $(X[m(i-1)+1, m(i)-1])_i$ is boundedly complete there is $z \in X$ with

$$\|z\| \leq \limsup \|R_{m(i)}\| \, \|x^{**} - w\| \leq K_\infty \|x^{**} - w\|$$

and $p_n(z) = p^{**}_n(x^{**} - w)$ for all n. Finally let $x = z + w$. We have $p_n(x) = p^{**}_n(x^{**})$ and

$$\|x\| = \|z + w\| < \|z\| + \varepsilon \leq K_\infty \|x^{**} - w\| + \varepsilon \leq K_\infty \|x^{**}\| + (K_\infty + 1)\varepsilon.$$

The choice of x is independent of ε and so $\|x\| \leq K_\infty \|x^{**}\|$. □

The following theorem summarizes our results for Banach space with PCP.

Theorem 6.3. *The following are equivalent for a Banach space X:*

(a) *X is separable and has* PCP.

(b) *Each norming* DDD (X_n) *of X can be blocked to be a boundedly complete* SBD.

(c) *Each norming* DDD (X_n) *of X can be blocked to be a 1-norming* JT*-decomposition in an equivalent norm.*

Proof. (c)⇒(b) is formal. (b)⇒(a) is in Bourgain and Rosenthal [3]. (a)⇒(b) is Theorem 5.2. (b)⇒(c) uses Proprosition 3.13 to equivalently renorm X so that the predecomposition space is 1-norming, Theorem 3.9 to block the DDD to a SBD with asymptotic constant 1 and finally Theorem 6.2 to show this blocking is a 1-norming JT-decompostion. □

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Department of Mathematics, Florida State University, Tallahassee, FL 32306-4510

E-mail address: **bellenot@math.fsu.edu**